

# Lecture 17

## Introduction to Feedback Control – part 2

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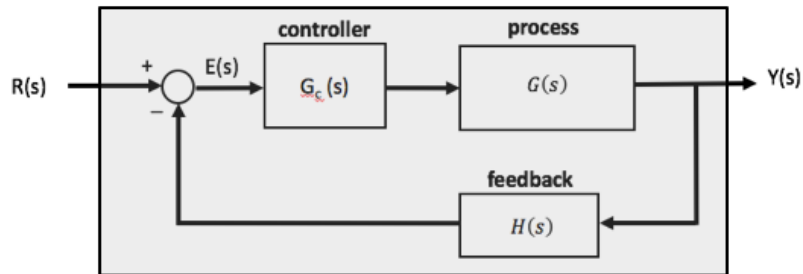


In part 2 of this introductory lecture to Feedback Control, we will look at how feedback changes the overall system transfer function. We will also examine how a system block diagram in the Laplace or s-domain can be simplified.

we will also examine how feedback control is used in a practical system. We will learn about how feedback give us many advantages. We will also take the DC motors we use in the mini-Segway as an example and see how feedback can help us to control the motor speed and its transient behaviour.

## A generic closed-loop control system

- ◆ Let us now consider a generic close loop system such as the motor or insulin pump control as shown here.



- ◆ The transfer function of the closed-loop control system from input  $R(s)$  to output  $Y(s)$  is (applying transforms 1 & 6):

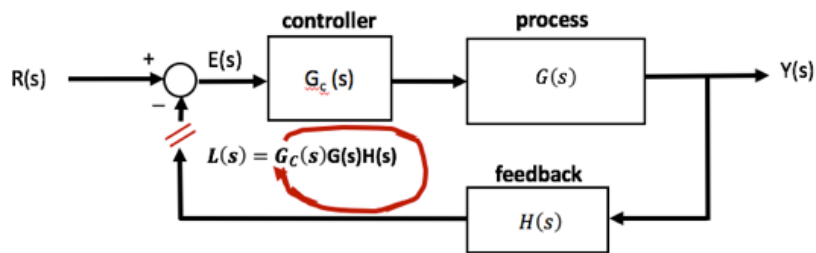
$$\frac{Y(s)}{R(s)} = \frac{G_c(s)G(s)}{1 + G_c(s)G(s)H(s)}$$

Let us apply these transformation technique to our closed-loop system of the motor or the insulin pump.

We can easily derive (using 1 and 6) that the transfer function from  $R(s)$  to  $Y(s)$  is as shown here.

We will see in the next lecture why this is significant.

## The concept of loop gain $L(s)$



- ◆ From the previous slide, we have the transfer function of a close-loop system as:

$$\frac{Y(s)}{R(s)} = \frac{G_c(s)G(s)}{1 + G_c(s)G(s)H(s)} = \frac{G_c(s)G(s)}{1 + L(s)}$$

- ◆ The quantity:  $L(s) = G_c(s)G(s)H(s)$  is known as **loop gain** of the system.
- ◆ It is the transfer function (gain) if you break the feedback loop at the point of feedback, and calculate the gain around the loop as shown.
- ◆ This quantity turns out to be most important in a feedback system because it affects many characteristics and behaviour in such a system.
- ◆ We will consider why such a close-loop system with feedback is beneficial in the next Lecture.

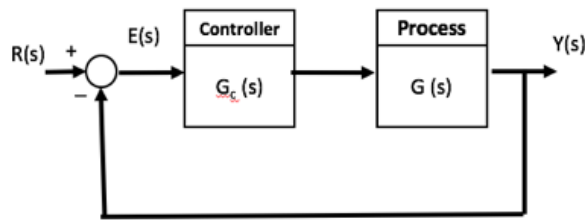
Loop gain is an important concept.

If we break the feedback loop at just BEFORE we subtract the feedback signal from the desired input  $R(s)$ , the gain of the system around the loop (without feedback) is known as **loop gain**. In the generic system shown here, the loop gain:

$$L(s) = G_c(s) G(s) H(s).$$

You will find this quantity popping up all over the place in any feedback systems. In many cases, the higher the loop gain, the “better” is the control system. However, increasing the loop gain also make the system more prone to **instability** (i.e. it can become oscillatory).

## Feedback makes system insensitive to G(s)



- ◆ Let us now assume that  $H(s) = 1$  to simplify things.
- ◆ We have seen from the last lecture that the transfer function of this closed-loop system is:

$$\frac{Y(s)}{R(s)} = \frac{G_c(s)G(s)}{1 + G_c(s)G(s)} = \frac{L(s)}{1 + L(s)}$$

- ◆ If  $L(s) = G_c(s)G(s) \gg 1$  then this term approaches 1!!
- ◆ In other words, the actual output  $Y(s)$  (e.g. motor speed) will track the desired input  $R(s)$  independent of  $G(s)$ , our system behaviour:

$$\frac{Y(s)}{R(s)} \approx 1 \quad \text{if} \quad G_c(s)G(s) \gg 1$$

Here is the feedback system we used in the last lecture. The controller is under our control. The purpose is that with the negative feedback, we can make the system behaviour in a better way than an open-loop system.

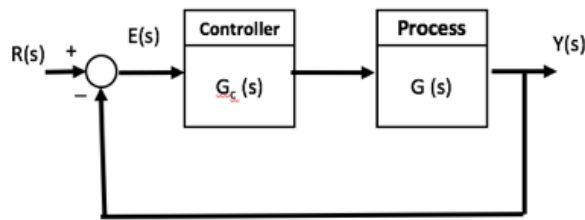
If you derive the transfer function from  $R(s)$  to  $Y(s)$ , you find that the transfer function of this closed-loop system becomes:

$$\frac{Y(s)}{R(s)} = \frac{G_c(s)G(s)}{1 + G_c(s)G(s)} = \frac{L(s)}{1 + L(s)}$$

Provided that the loop gain  $L(s)$  is large compared to 1, this gives us a value of 1 (for all  $s$  values). That means the output  $Y(s)$  is tracking the set point (desired control variable value)  $R(s)$  because of this feedback loop.

This is an important result. The closed-loop system behaviour is now **INDEPENDENT** of  $G(s)$ , the system we are controlling, as long as the loop gain  $L(s) = G_c(s)G(s)$  is large as compared to 1.

## Feedback yields small steady-state error e(t)



- ◆ Let us suppose the input to the system is a step at  $t=0$  with a magnitude of  $A$ :  $r(t) = Au(t)$ .
- ◆ Then  $R(s) = A \frac{1}{s}$  (because Laplace transform of  $u(t)$  is  $1/s$ )
- ◆ We know that in this system,  $y(t)$  will track  $r(t)$  from the previous two slides. The question is: **“After transient has died down, what is error  $e(t)$ ?”**
- ◆ To calculate this steady-state error, we need to use the **final-value theorem**, which states:

$$\lim_{t \rightarrow \infty} e(t) = \lim_{s \rightarrow 0} sE(s)$$

- ◆ Therefore,

$$\lim_{t \rightarrow \infty} e(t) = \lim_{s \rightarrow 0} sE(s) = \lim_{s \rightarrow 0} s \frac{1}{1 + L(s)} A \frac{1}{s} = \frac{A}{1 + L(0)}$$

- ◆ So the steady-state error is reduced by a factor of  $(1 + L(0))$

Now let us consider the error  $e(t)$  or  $E(s)$  in the  $s$ -domain. Ideally you want  $Y(s)$  to be exactly  $R(s)$ , that is, what we set as desired, is what we get. However, in any systems, there may be an error.

$$E(s) = R(s) - Y(s).$$

Let us now consider the case that the input  $r(t)$  is a step function with a step value of  $A$ , i.e.  $r(t) = A u(t)$  at  $t = 0$ .

This is modelling the case that you may be controlling a robot arm to move from one point to any other at time = 0, or changing the motor speed from one to another.

Remember from earlier lecture that the Laplace transform of  $A u(t)$  is:

$$R(s) = L\{Au(t)\} = A \frac{1}{s}$$

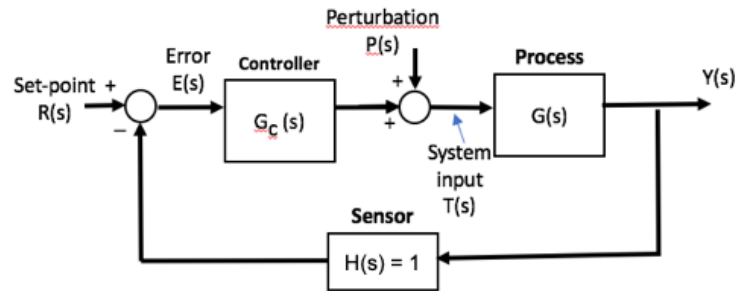
The question is: after such as step, what will the output eventually settle down to? In order to answer this question, we have to use some important theorem, known as the final-value theorem, which states:

$$\lim_{t \rightarrow \infty} e(t) = \lim_{s \rightarrow 0} sE(s)$$

I don't want to prove this, but it allows you to compute the final value of a signal using only the  $s$ -domain equations (i.e. the transfer function). As shown in the slide here, the final value of the error  $e(t) = A / (1 + L(0))$ , where  $L(0)$  is the loop gain at  $s=0$ , or the DC gain of  $L(s)$ .

In other words, the steady-state error (when  $s=0$ ) is  $A$  reduced by a factor of  $(1+L(0))$

## Feedback reduces impact of perturbations



- ◆ Let us put back the perturbation  $p(t)$  to the system.
- ◆ Assume  $R(s) = 0$ , and the effect of perturbation  $P(s)$  on output  $Y(s)$  can be found by considering the expression for  $T(s)$  at the input to our system under control:

$$T(s) = P(s) - T(s)G(s)G_C(s)$$

- ◆ In open-loop,  $Y(s) = G(s)P(s)$

$$\Rightarrow T(s) = \frac{1}{1 + L(s)} P(s) = \frac{Y(s)}{G(s)}$$

$$\Rightarrow Y(s) = \frac{G(s)}{1 + L(s)} P(s)$$

- ◆ In closed-loop, the disturbance is reduced by the factor:  $\frac{1}{1 + L(s)}$

Let us now consider the impact of closed-loop on perturbations  $P(s)$  or  $p(t)$ .

This is added as shown in the block diagram.

We are interested in the relationship between  $Y(s)$ , the output, and the perturbation  $P(s)$ .

To work this out, there is a good trick – always evaluate at the signal point AFTER the summer. In this case, it is the input to the system  $G(s)$ . Let us call this  $T(s)$ .

Some simple algebra yield the answer:

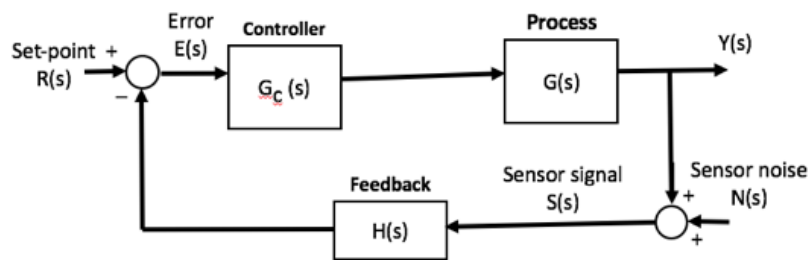
$$Y(s) = \frac{G(s)}{1 + L(s)} P(s)$$

If we did not have closed-loop, but have a simple open-loop system, then  $Y(s) = G(s)P(s)$ . That is, the perturbation is passed to the output through our system as-is.

However, putting the system  $G(s)$  in this feedback loop, we reduce this effect by the factor  $\frac{1}{1 + L(s)}$ . Note that this factor comes up all the time!

If loop gain  $L(s)$  is relatively large as compared to 1, then the perturbation is reduced by the factor of  $L(s)$  at all value of  $s$ .

## Feedback introduces problem with sensor noise



- ◆ Let us put back the sensor noise  $n(t)$  to the system.
- ◆ Assume  $R(s) = 0$ , and the effect of  $N(s)$  on  $Y(s)$  can be found by considering the expression for  $S(s)$ , the sensor signal in the feedback path:
 
$$S(s) = N(s) - H(s)G_c(s)G(s)S(s)$$
- ◆ In open-loop, sensor is not an issue.
 
$$\Rightarrow S(s) = \frac{1}{1 + L(s)}N(s)$$

$$\Rightarrow Y(s) = -L(s)S(s) = -\frac{L(s)}{1 + L(s)}N(s)$$
- ◆ In closed-loop, we want  $L(s)$  to be small in order to have good attenuation of the sensor noise.
- ◆ This is in contradiction to the previous two properties. (We will consider this in more details later.)

Now let us put into the system the sensor noise  $n(t)$  or  $N(s)$ .  
With some manipulation, we found that:

$$Y(s) = -L(s)S(s) = -\frac{L(s)}{1 + L(s)}N(s)$$

Unfortunately the output is VERY MUCH affected by the noise. If  $L(s)$  is large, as we have assumed in previous slides, then  $N(s)$  is more or less passed to the output  $Y(s)$ .

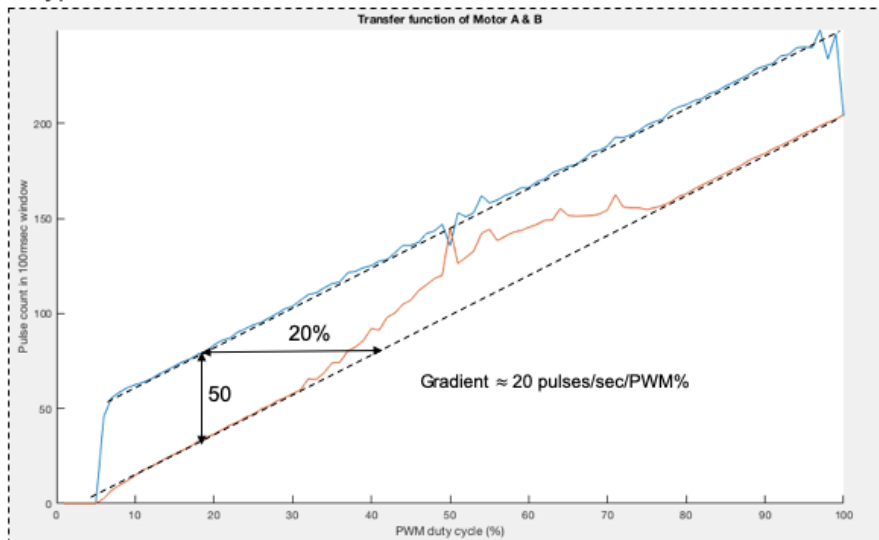
Fortunately, in many practical systems,  $N(s)$ , the noise, generally is at high frequency. If  $G(s)$  or  $G_c(s)$  has low gain at high frequency (lowpass filter),  $L(s) \ll 1$  at high  $s$ , then,

$$Y(s) \approx -L(s)N(s)$$

Since  $L(s) \ll 1$ ,  $N(s)$  can be suppressed.

## Practical process - Our DC Motors

- ◆ The two DC motors we use on the Segway may have very different characteristics.
- ◆ Here are plots of motor speed (in number of pulses per 100msec) vs PWM duty cycle for two typical motors:



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So far, we have been considering the theoretical foundation of a feedback system. Let us now consider something that is both real and that you are familiar with.

Plotted here is the speed of two typical DC motors used for our project. These are motors from the mini-Segway that I have been personally using. The plot is the pulse count from the hall effect sensor (in a 100ms window) vs the PWM duty cycle driving each motor.

We can make the following observations:

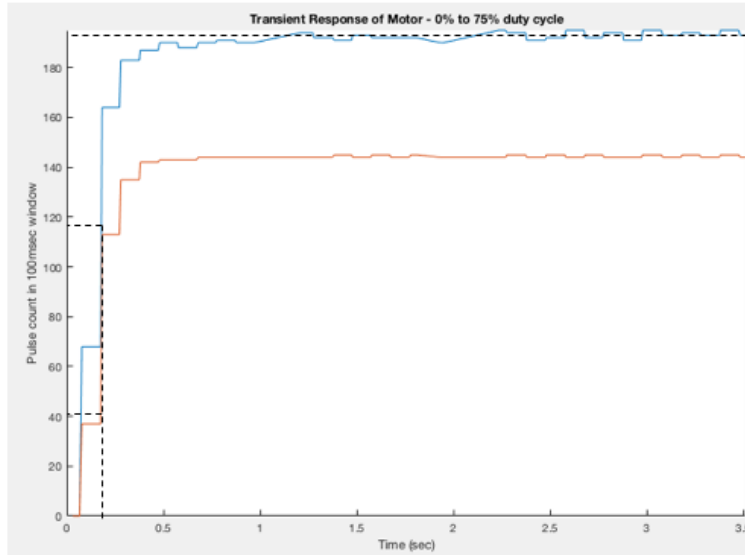
1. The characteristic is relatively linear – that is speed is proportional to PWM value.
2. The two motors have more or less the same gradient of around 20 pulses/sec/PWM%. In other words, if you increase PWM value by 10, you can expect the pulse count to increase by 20 in the duration of 1sec.
3. The two motors are not matched. The BLUE motor is consistently faster (offset) from the RED motor by 50 pulses per second. Or in order for the two motor to go at the same speed, the RED motor needs to be driven with an extra 20% in PWM!
4. Both motors do not start turning unless the PWM value exceeds around 6 or 7%.
5. Something funny happens to the RED motor in mid-range. Not sure why!

This characteristic really demonstrates why we may need to use feedback control in order to make the motor drive at desired speed independent of its own characteristics.



## Step response of the motor

- ◆ Here is the plot of the step response of two typical motors.
- ◆ The time constant (time it takes to reach 63% of final speed) is around 0.2sec.



I also test the response of the motors to a step input (going from PWM=0% to 75%).

This shows that both motor behaves approximately to a first order function with an exponential rise of time-constant = 0.2sec.

Any first order system responses to a step inout as a exponent rising signal with a time constant  $\tau$ , where  $\tau$  is the time it takes to reach 63% of final value. (Remember the RC time constant from Year 1?)

## Model of the motor – G(s)

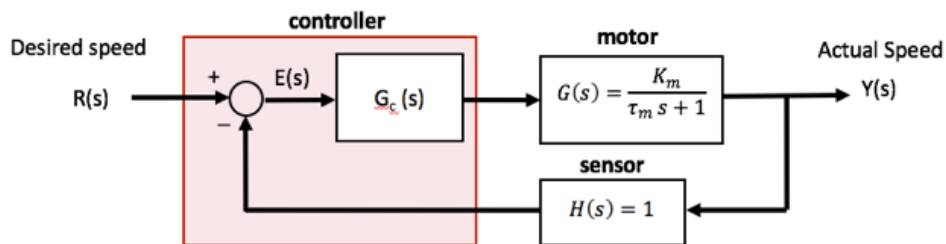
- ◆ We can model the motor as having a transfer function:

$$G(s) = \frac{K_m}{\tau_m s + 1}$$

- ◆  $K_m$  is the dc gain, which is the gradient of the plot in slide 6 (i.e. the gain of the system when  $s = 0$ , or steady-state). Therefore  $K_m = 20$  pulses/sec/PWM%
- ◆  $\tau_m$  is the time constant of the motor, which is estimated to be around 0.2sec in slide 7.
- ◆ Therefore:

$$G(s) = \frac{20}{0.2s + 1}$$

- ◆ Assuming  $H(s) = 1$ , we now put this motor in a feedback loop with a controller  $G_c(s)$ .



With these information, we can write down the transfer function of the motor. The dc gain is  $K_m$ , and it is 20 pulses/sec/PWM%. The time constant is 0.2sec.

Therefore the motor transfer function (from PWM duty cycle in %) to output speed (in pulses/sec) is:

$$G(s) = \frac{20}{0.2s + 1}$$

In this discussion, we assume the feedback transfer  $H(s) = 1$ .

## Proportional feedback

◆ Let us start with a simple controller with  $G_c(s) = K_p$ , where  $K_p$  is a constant.

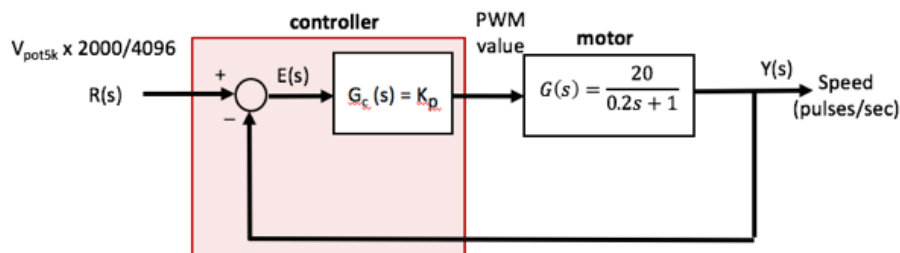
◆ From transforms 1 & 6, we get: 
$$\frac{Y(s)}{R(s)} = \frac{L(s)}{1 + L(s)} = \frac{K_p \frac{20}{0.2s + 1}}{1 + K_p \frac{20}{0.2s + 1}}$$

◆ Therefore the closed-loop transfer function is:

$$\frac{Y(s)}{R(s)} = \frac{20K_p}{1 + 20K_p + 0.2s} = \frac{20K_p/(1 + 20K_p)}{1 + \left(\frac{0.2}{1 + 20K_p}\right)s} = \frac{K_C}{1 + \tau_c s}$$

$$K_C = \frac{20K_p}{1 + 20K_p}$$

$$\tau_c = \left(\frac{0.2}{1 + 20K_p}\right)$$



Now let us plug this motor into our simple feedback system with the control  $G_c(s)$ . Furthermore, let us assume that the controller simply multiply the error signal  $e(t)$  by a constant gain  $K_p$ . In other words, the drive signal is proportional to the error signal.

This is known as “**proportional control**” and the proportional gain is  $K_p$ . Some algebraic manipulation give us:

$$\frac{Y(s)}{R(s)} = \frac{20K_p}{1 + 20K_p + 0.2s} = \frac{20K_p/(1 + 20K_p)}{1 + \left(\frac{0.2}{1 + 20K_p}\right)s} = \frac{K_C}{1 + \tau_c s}$$

$$K_C = \frac{20K_p}{1 + 20K_p} \quad \tau_c = \left(\frac{0.2}{1 + 20K_p}\right)$$

We will consider the implication of this result next.

## How are things improved with proportional feedback?

- ◆ For our system, loop gain is  $L(s) = 20K_p$  for  $s=0$ . Assuming  $K_p = 5$ , we get a steady-state gain of:

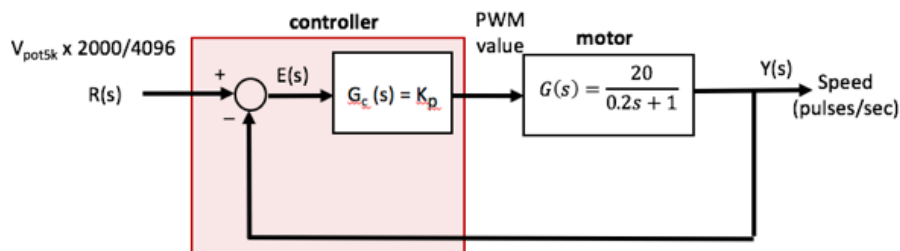
$$\left. \frac{Y(s)}{R(s)} \right|_{s=0} = \left. \frac{L(s)}{1 + L(s)} \right|_{s=0} = \frac{20K_p}{1 + 20K_p} = \frac{100}{101} = 0.99$$

- ◆ The steady-state error for a step input of magnitude  $A$  (i.e.  $A * u(t)$ ) is:

$$E(s) \Big|_{s=0} = \frac{1}{1 + L(s)} \Big|_{s=0} A = \frac{1}{1 + L(0)} A = 0.01A$$

- ◆ Perturbation is also reduced by this factor (see slide 6):

$$Y(s) = 0.01P(s)$$



Remember: 
$$\frac{Y(s)}{R(s)} = \frac{20K_p}{1 + 20K_p + 0.2s} = \frac{20K_p / (1 + 20K_p)}{1 + \left( \frac{0.2}{1 + 20K_p} \right) s} = \frac{K_C}{1 + \tau_c s}$$

$$K_C = \frac{20K_p}{1 + 20K_p} \quad \tau_c = \left( \frac{0.2}{1 + 20K_p} \right)$$

Let us assume the proportional gain  $K_p = 5$  (not unreasonable), then the steady-state transfer function:

$$Y(0) = 0.99 R(0)$$

In other words, at steady state, the output tracks the input to 1%.

Also the steady-state error for a step of magnitude  $A$ , is also 1% of  $A$  – very small.

The sensitivity to perturbation is also reduced by a factor of 100.

Finally, if you rearrange to closed-loop transfer function to the form shown, you can see that the closed-loop time constant  $\tau_c$  is also reduced from 0.2 to around 0.002, also around 100 times.

This is the wonderful world of feedback control!

One word of warning – everything is not as rosy as it may seem. We have not considered an important factor: the stability of the system. As we increase the proportional gain  $K_p$ , the system could go unstable if the system is NOT strictly first order. We will consider stability issue later.